Adaptive and Generic Parallel Exact Linear Algebra

Ziad SULTAN

Université Grenoble Alpes
PhD Defense
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Mme Laura Grigori (Referee)       M. Pascal Giorgi (Examiner)
M. Arne Stojohann (Referee)       M. Denis Trystram (Examiner)
M. Jean-Guillaume Dumas (Director) M. Clément Pernet (Co-Director)
PhD research context

- **Parallel computing**

Symmetric Multi-Processors
PhD research context

- Parallel computing

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Symmetric Multi-Processors

- Effective exact parallel linear algebra
  - Solve target problems: dedicated codes
  - Widely distributed software: general purpose codes (SAGE, Macauley2)
PhD research context

- **Parallel computing**

Symmetric Multi-Processors

- **Effective exact parallel linear algebra**
  - Solve target problems: dedicated codes
  - Widely distributed software: general purpose codes (SAGE, Macauley2)

- **Design a software for parallel exact linear algebra**
Exact linear algebra

Exact computation

- Computation in computer algebra
  \[ \rightarrow \text{computing exactly: over } \mathbb{Z}, \mathbb{Q}, \mathbb{Z}[x] \]
- In practice, often boils down to computation over prime fields \( \mathbb{Z}/p\mathbb{Z} \)
**Exact linear algebra**

### Exact computation
- Computation in computer algebra
  - computing exactly: over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}[x]$
- In practice, often boils down to computation over prime fields $\mathbb{Z}/p\mathbb{Z}$

### Exact linear algebra applications
- Breaking Discrete Log Pb. in quasi-polynomial time [Barbulescu & al. 14]
- Building modular form databases to test the BSD conjecture [Stein 12]
- Exact mixed-integer programming [Steffy et al. 12]
- Formal verification of Hales proof of Kepler conjecture [Hales 05]
Use case example of an application

HPAC on-going Challenge: D.L.P. cryptanalysis over curves over $\mathbb{F}(2^{29})$.

**Problem dimensions**

- Sparse matrix with 126M var. / 130M eq.
- Modulo a prime number on 114 bits: 20769187434139310549529495610151239
- Matrix has 520M non-zero
Use case example of an application

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**Main steps of block Wiedemann**

- First filtering (structured Gauss)
  → $n$Rows: 8.7M, $n$Cols: 8.7M.
  → Matrix has 810M non-zero with blocks $32 \times 16$
- MinPoly coefficients $16 \times 16$, degree 545966
  → needs efficient PLUQ factorization!
- Evaluation uses M.M. : $(n \times 32)$ times $(32 \times 32)$ → $n$ is large!
Dense exact linear algebra

Dense linear algebra: A key building block for:
- dense problems by nature (Hermite-Padé approx, ...)
- Sparse problems degenerate to dense:
  - Sparse Direct:
    Switch to dense after fill-in
  - Sparse Iterative:
    Induce dense elimination on blocks of iterated vectors
    (block-Wiedemann, block Lanczos, ...)
Gaussian elimination in exact dense algebra

Gaussian elimination is a building block in dense linear algebra

**Matrix factorization** (LU decomposition)
- Solving linear systems
- Computing determinant
- Rank.

**Linear dependencies** (Echelon structure)
- Characteristic Polynomial: Finding Krylov basis [Keller Gehrig 85]
- Grobner basis computation: F4 algorithm [FGB]
### Design of parallel dense exact linear algebra

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**Parallelizing dense linear algebra**

- Specificities of exact linear algebra
  - Recursive algorithms
  - Rank deficiencies

- Similarities with numerical linear algebra
  Parallel blocking is constrained by pivoting:

  **Numerical**: ensuring numerical stability
  **Exact**: recovering rank profiles and echelon structure
Outline

1. Pivoting and rank profiles
2. Generic parallel Linear Algebra
3. Parallel exact Gaussian elimination
Outline

1. Pivoting and rank profiles
2. Generic parallel Linear Algebra
3. Parallel exact Gaussian elimination
### Linear dependencies and row/column rank profiles

**Definition** (Row Rank Profile : RowRP)

Given $A \in K^{m \times n}$, $r = \text{rank}(A)$.

- **Informally**: first $r$ linearly independent rows
- **Formally**: lexicominimal sub-sequence of $(1, \ldots, m)$ of $r$ indices of linearly independant rows.

**Example**

$$
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Rank = 3

RowRP = $\{1, 2, 4\}$

ColRP = $\{1, 2, 3\}$
Linear dependencies and row/column rank profiles

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Rank = 3

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Linear dependencies and row/column rank profiles

**Definition (Column Rank Profile: ColRP)**

Given $A \in K^{m \times n}$, $r = \text{rank}(A)$.

- **informally**: first $r$ linearly independent columns
- **formally**: lexicominimal sub-sequence of $(1, \ldots, m)$ of $r$ indices of linearly independant columns.

**Example**

$$\begin{bmatrix}
1 & 1 & 0 & 0 \\
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- Rank = 3
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\textbf{Generic} RowRP/ColRP: if it equals \{1, \ldots, r\}.
Linear dependencies and row/column rank profiles

**Definition (Column Rank Profile: ColRP)**

Given $A \in K^{m \times n}$, $r = \text{rank}(A)$.

Informally: *first* $r$ linearly independent columns

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*Rank* = 3

*RowRP* = \{1,2,4\}

*ColRP* = \{1,2,3\} $\rightarrow$ Generic ColRP.

**Generic** RowRP/ColRP: if it equals \{1, \ldots, r\}.
Motivation

Pivoting and rank profiles

Generic parallel LinAlg

Parallel Exact Gaussian elimination

Linear dependencies

Computing rank profiles

Via Gaussian elimination revealing row echelon forms:

[Ibarra, Moran and Hui 82]

[Keller-Gehrig 85]

[Storjohann 00]

[Jeannerod, Pernet and Storjohann 13]
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Lessons learned (or what we thought was necessary):

- treat rows in order
- exhaust all columns before next row
- **slab** block splitting (rec or iter)
  
  => similar to partial pivoting
Motivation

Need more flexible blocking

Slab blocking
- can lead to inefficient memory access patterns
- is harder to parallelize

Tile blocking instead?
Motivation

Need more flexible blocking

Slab blocking

- can lead to inefficient memory access patterns
- is harder to parallelize

Tile blocking instead?

Gathering linear independence invariants

Two ways to look at a matrix (looking left or right):

- Row rank profile, column echelon form
- Column rank profile, row echelon form

Unique invariant?
The rank profile Matrix

**Theorem**

Let $A \in F^{m \times n}$.

There exists a unique, $m \times n$, rank($A$)-sub-permutation matrix $R_A$ of which every leading sub-matrix has the same rank as the corresponding leading sub-matrix of $A$. 

$$R_A$$
Linear dependencies

The rank profile Matrix

Theorem

Let $A \in \mathbb{F}^{m \times n}$. There exists a unique, $m \times n$, rank($A$)-sub-permutation matrix $R_A$ of which every leading sub-matrix has the same rank as the corresponding leading sub-matrix of $A$.

Example

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
1 & 3 & 2 & 0 \\
2 & 5 & 4 & 7 \\
\end{bmatrix}
\quad \begin{bmatrix}
0 & 1 & 0 & 0 \\
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Let $A \in \mathbb{F}^{m \times n}$.
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Properties of the rank profile matrix

Particular cases

- \( A \) invertible \( \iff \mathcal{R}_A \) is a permutation
- \( A \) is square with generic rank profile \( \iff \mathcal{R}_A = I_n \)
Linear dependencies

Properties of the rank profile matrix

Particular cases
- $A$ invertible $\iff \mathcal{R}_A$ is a permutation
- $A$ is square with generic rank profile $\iff \mathcal{R}_A = I_n$

Properties
- $\mathcal{R}_A$ encodes the $RowRP(A)$ and the $ColRP(A)$
- All leading rank profiles
- $\mathcal{R}_A$ is unique $\implies$ new normal form.
Linear dependencies

When does a PLUQ decomposition reveal the rank profile matrix?

Focus on the pivoting strategy:
- Pivot search:
  - finding a pivot with minimal coordinates
- Permutation to bring the pivot to the main diagonal
Pivoting and permutation strategies

Pivot Search

Pivot’s \((i, j)\) position minimizes some pre-order:

Row order: any non-zero on the first non-zero row
Pivoting and permutation strategies

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Pivot's \((i, j)\) position minimizes some pre-order:

**Row/Col order**: any non-zero on the first non-zero row/col
Pivoting and permutation strategies

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Pivot’s \((i, j)\) position minimizes some pre-order:

- **Row/Col order**: any non-zero on the first non-zero row/col
- **Lex order**: first non-zero on the first non-zero row
Pivoting and permutation strategies

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Computing the Rank Profile Matrix

Pivoting and permutation strategies

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**Product order:** first non-zero in the \((i, j)\) leading sub-matrix
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Permutation

- Transpositions
Pivoting and permutation strategies

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**Permutation**

- Transpositions
- Cyclic Rotations

Cyclic rotation
## Computing the Rank Profile Matrix

### Pivoting strategies revealing rank profiles

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**Motivation**

Pivoting and rank profiles

**Generic parallel LinAlg**

Parallel Exact Gaussian elimination

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<td>[KG85] [JPS13]</td>
</tr>
<tr>
<td>Lexico.</td>
<td>Transposition</td>
<td>Transposition</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[Sto00]</td>
</tr>
<tr>
<td>Lexico.</td>
<td>Transposition</td>
<td>Rotation</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[DPS15]</td>
</tr>
<tr>
<td>Lexico.</td>
<td>Rotation</td>
<td>Transposition</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[DPS15]</td>
</tr>
<tr>
<td>Rev. lex.</td>
<td>Transposition</td>
<td>Transposition</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[Sto00]</td>
</tr>
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<td>Rev. lex.</td>
<td>Rotation</td>
<td>Transposition</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[DPS15]</td>
</tr>
<tr>
<td>Rev. lex.</td>
<td>Rotation</td>
<td>Transposition</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[DPS15]</td>
</tr>
<tr>
<td>Product</td>
<td>Rotation</td>
<td>Transposition</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[DPS15]</td>
</tr>
<tr>
<td>Product</td>
<td>Transposition</td>
<td>Rotation</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[DPS15]</td>
</tr>
<tr>
<td>Product</td>
<td>Rotation</td>
<td>Rotation</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[DPS13]</td>
</tr>
</tbody>
</table>

$P, L, U, Q \leftarrow PLUQ(A)$ and $P \begin{bmatrix} I_r \\ 0 \end{bmatrix} Q = \mathcal{R}_A.$
Echelon forms

\[ \mathcal{R}_A = P \begin{bmatrix} L \cdot P_s & 0_{m \times (n-r)} \end{bmatrix} \]

\[ C = P \begin{bmatrix} L \cdot P_s & 0_{m \times (n-r)} \end{bmatrix} \]

\[ E = \begin{bmatrix} Q_s \cdot U \\ 0_{(n-r) \times n} \end{bmatrix} Q \]

\[ Q_s \cdot U \]
Echelon forms

\[ \mathcal{R}_A = P \cdot L \cdot U \cdot Q \]

for

\[ C = PLP_s \]

sort

\[ Q_s U Q = E \]

\[ C = P \left[ L \cdot P_s \quad 0_{m \times (n-r)} \right], \quad F = P_s^T Q_s^T, \quad E = \left[ \begin{array}{c} Q_s \cdot U \\ 0_{(n-r) \times n} \end{array} \right] Q \]

Bonus: Generalized Bruhat CFE.
Tile recursive PLUQ algorithm

2 × 2 block splitting
Tile recursive PLUQ algorithm

Recursive call
Tile recursive PLUQ algorithm

\[ \text{TRSM} : B \leftarrow BU^{-1} \]
Tile recursive PLUQ algorithm

\[ \text{TRSM} : B \leftarrow L^{-1}B \]
Tile recursive PLUQ algorithm

\[
\text{fgemm : } C \leftarrow C - A \times B
\]
Tile recursive PLUQ algorithm

\[
\text{fgemm : } C \leftarrow C - A \times B
\]
Tile recursive PLUQ algorithm

\[ \text{fgemm}: C \leftarrow C - A \times B \]
Tile recursive PLUQ algorithm

2 independent recursive calls (product order search)
Tile recursive PLUQ algorithm

\[ \text{TRSM} : B \leftarrow BU^{-1} \]
Tile recursive PLUQ algorithm

\[
\text{TRSM} : B \leftarrow L^{-1} B
\]
Tile recursive PLUQ algorithm

\[ \text{fgemm} : C \leftarrow C - A \times B \]
fgemm: $C \leftarrow C - A \times B$
Tile recursive PLUQ algorithm

\[ \text{fgemm} : C \gets C - A \times B \]
Tile recursive PLUQ algorithm

Recursive call
Tile recursive PLUQ algorithm

Puzzle game (block permutations)
New PLUQ algorithm

- New state of the art algo that computes faster PLUQ decomposition
- Computes more information (the rank profile matrix $\mathcal{R}_A$)
New PLUQ algorithm

- New state of the art algo that computes faster PLUQ decomposition
- Computes more information (the rank profile matrix \( \mathcal{R}_A \))

![Graph showing effective Gflops for different algorithms and matrix sizes](image.png)
New PLUQ algorithm

- New state of the art algo that computes faster PLUQ decomposition
- Computes more information (the rank profile matrix $R_A$)
New PLUQ algorithm

- New state of the art algo that computes faster PLUQ decomposition
- Computes more information (the rank profile matrix $\mathcal{R}_A$)

Execution on 1 core (3.5GHz) → effective 31 Gfops (AVX2 + sub-cubic complexity)
Outline

1. Pivoting and rank profiles
2. Generic parallel Linear Algebra
3. Parallel exact Gaussian elimination
FFLAS-FFPACK library

FFLAS-FFPACK features

- High performance implementation of basic linear algebra routines over word size prime fields
- Exact alternative to the numerical BLAS library
- Exact triangularization, Sys. solving, Det, Inv., CharPoly

Parallel FFLAS-FFPACK

Explore:

- several algorithms and variants
- parallel runtimes and languages:
  - unified parallel language harnessing different runtimes (OMP, TBB, xKaapi, ...)
  - Abstraction for the user
- data parallelism vs task parallelism
Parallel computation constraints: exact and numeric

In state of the art numerical libraries:

- Often non-singular matrices with fixed static cutting.
  → easier to manually map and schedule tasks or threads.
- Use of iterative algorithms → often one or two levels of parallelism.
Parallel computation constraints: exact and numeric

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Our experience in exact linear algebra:

- Sub-cubic complexity: \( O(n^\omega) \) [Strassen]
  → Coarser grain cutting
  → Recursive algorithms.
  → Parallel runtime system that implements well recursive tasks.
- Rank deficiencies → tasks of unbalanced workloads.
- Recursion and code composition → multiple levels of parallelism.
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→ Need for a high level parallel programming environments
Parallelization of FFLAS-FFPACK library

Requirements of high level parallel programming environments

**Features required**

Portability, Performance and scalability. But more precisely:

- Runtime system with good performance for recursive tasks.
- Handle efficiently unbalanced workloads.
- Efficient range cutting for parallel for.
Parallelization of FFLAS-FFPACK library

Requirements of high level parallel programming environments

Features required

Portability, Performance and scalability. But more precisely:

- Runtime system with good performance for recursive tasks.
- Handle efficiently unbalanced workloads.
- Efficient range cutting for parallel for.

No parallel environment offers all these features

→ Need to design a code independently from the runtime system
→ Using runtime systems as a plugin
Runtime systems to be supported

OpenMP 3.x and 4.0 supported directives: (using libgomp)
- Data sharing attributes:
  - OMP3 `shared`: data visible and accessible by all threads
  - OMP3 `firstprivate`: local copy of original value
  - OMP4 `depend`: set data dependencies
- Synchronization clauses: `#pragma omp taskwait`

xKaapi: via the libkomp [BDG12] library:
- OpenMP directives → xKaapi tasks.
- Re-implem. of task handling and management.
- Better recursive tasks execution.

TBB: designed for nested and recursive parallelism
- `parallel_for`
- `tbb::task_group, wait(), run()` using C++11 lambda functions
Parallel Algebraic Linear Algebra Dedicated Interface

Mainly macro-based keywords
- No function call runtime overhead when using macros.
- No important modifications to be done to original program.
- Macros can be used also for C-based libraries.

Complementary C++ template functions
- Implement the different cutting strategies.
- Store the iterators
Data parallelism: SPMD programming

- Parallel region: chunks are dispatched on multiple proc.
- Supported: PARFOR1D, PARFOR2D, PARFORBLOCK1D, PARFORBLOCK2D.

Example: Loop Summing in C++

```cpp
for(size_t i=0; i<n; ++i){
    T[i] = T1[i] + T2[i];
}
```

Example: Loop Summing in OpenMP

```cpp
#pragma omp parallel for
for(size_t i=0; i<n; ++i){
    T[i] = T1[i] + T2[i];
}
```

Example: Loop Summing in PALADIn

```cpp
PARFOR1D( i, n, SPLITTER(),
    T[i] = T1[i] + T2[i];
);
```

→ The SPLITTER keyword sets the cutting strategy.
Iterative Cutting Strategies 1D

Splitting over one dimension

- \texttt{SPLITTER}(p, \text{THREADS}) : \( p \) partitions = \#tasks
- \texttt{SPLITTER}(p, \text{GRAIN}) : BlockSize : \( BS = p \)
- \texttt{SPLITTER}(p, \text{FIXED}) : BlockSize : \( BS = 256 \)
- \texttt{SPLITTER}(p) : \( p \) tasks with default strategy (\text{THREADS})
- \texttt{SPLITTER}() : default strategy with \( p = \# \) available processors

Code example: Matrix add in parallel

```c
void pfadd(const Field & F, const Element *A, const Element *B, Element *C, size_t n)
{
    PARFORBLOCK1D(it, n, SPLITTER(32, THREADS),
    FFLAS::fadd(F, it.end() - it.begin(), n,
                A+ it.begin() * n, n,
                B+ it.begin() * n, n,
                C+ it.begin() * n, n);
}
```
Iterative cutting strategies 2D

Data parallelism: SPLITTER keyword

- \text{SPLITTER}(p, \text{ROW}, \text{THREADS}): p\text{ row blocks}
- \text{SPLITTER}(p, \text{ROW}, \text{FIXED}): \text{row } BS = 256
- \text{SPLITTER}(p, \text{ROW}, \text{GRAIN}): \text{row } BS = p
Iterative cutting strategies 2D

Data parallelism : SPLITTER keyword

- \text{SPLITTER}(p, \text{ROW, THREADS}) : p \text{ row blocks}
- \text{SPLITTER}(p, \text{ROW, FIXED}) : \text{row BS} = 256
- \text{SPLITTER}(p, \text{ROW, GRAIN}) : \text{row BS} = p
- \text{SPLITTER}(p, \text{COLUMN, THREADS}) : p \text{ col blocks}
- \text{SPLITTER}(p, \text{COLUMN, FIXED}) : \text{col BS} = 256
- \text{SPLITTER}(p, \text{COLUMN, GRAIN}) : \text{col BS} = p
Parallelization of FFLAS-FFPACK library

Iterative cutting strategies 2D

Data parallelism : SPLITTER keyword

- SPLITTER \((p, \text{ROW, THREADS})\) : \(p\) row blocks
- SPLITTER \((p, \text{ROW, FIXED})\) : row \(BS = 256\)
- SPLITTER \((p, \text{ROW, GRAIN})\) : row \(BS = p\)
- SPLITTER \((p, \text{COLUMN, THREADS})\) : \(p\) col blocks
- SPLITTER \((p, \text{COLUMN, FIXED})\) : col \(BS = 256\)
- SPLITTER \((p, \text{COLUMN, GRAIN})\) : col \(BS = p\)
- SPLITTER \((p, \text{BLOCK, THREADS})\) : \(s \times t\) blocks
- SPLITTER \((p, \text{BLOCK, FIXED})\) : \(BS = 256\)
- SPLITTER \((p, \text{BLOCK, GRAIN})\) : \(BS = p\)
Iterative cutting strategies 2D

Data parallelism: SPLITTER keyword

- \texttt{SPLITTER}(p, \texttt{ROW}, \texttt{THREADS}): p row blocks
- \texttt{SPLITTER}(p, \texttt{ROW}, \texttt{FIXED}): row \texttt{BS} = 256
- \texttt{SPLITTER}(p, \texttt{ROW}, \texttt{GRAIN}): row \texttt{BS} = p
- \texttt{SPLITTER}(p, \texttt{COLUMN}, \texttt{THREADS}): p col blocks
- \texttt{SPLITTER}(p, \texttt{COLUMN}, \texttt{FIXED}): col \texttt{BS} = 256
- \texttt{SPLITTER}(p, \texttt{COLUMN}, \texttt{GRAIN}): col \texttt{BS} = p
- \texttt{SPLITTER}(p, \texttt{BLOCK}, \texttt{THREADS}): s \times t blocks
- \texttt{SPLITTER}(p, \texttt{BLOCK}, \texttt{FIXED}): \texttt{BS} = 256
- \texttt{SPLITTER}(p, \texttt{BLOCK}, \texttt{GRAIN}): \texttt{BS} = p
- \texttt{NOSPLIT}(): \texttt{sequential execution}
Iterative cutting strategies 2D

Data parallelism: SPLITTER keyword

- **SPLITTER**($p$, ROW, THREADS): $p$ row blocks
- **SPLITTER**($p$, ROW, FIXED): row $BS = 256$
- **SPLITTER**($p$, ROW, GRAIN): row $BS = p$
- **SPLITTER**($p$, COLUMN, THREADS): $p$ col blocks
- **SPLITTER**($p$, COLUMN, FIXED): col $BS = 256$
- **SPLITTER**($p$, COLUMN, GRAIN): col $BS = p$
- **SPLITTER**($p$, BLOCK, THREADS): $s \times t$ blocks
- **SPLITTER**($p$, BLOCK, FIXED): $BS = 256$
- **SPLITTER**($p$, BLOCK, GRAIN): $BS = p$

**NOSPLIT()**: sequential execution

```c
PARFORBLOCK2D(iter, m, n, SPLITTER(),

    fgemm(..., A + iter.ibegin() * lda, lda,
    B + iter.jbegin(), ldb, beta,
    C + iter.ibegin() * ldc + iter.jbegin(), ldc);
```

Parallelization of FFLAS-FFPACK library
Iterative cutting strategies 2D

Data parallelism: splittter keyword

- **SPLITTER**(p, ROW, THREADS): p row blocks
- **SPLITTER**(p, ROW, FIXED): row BS = 256
- **SPLITTER**(p, ROW, GRAIN): row BS = p
- **SPLITTER**(p, COLUMN, THREADS): p col blocks
- **SPLITTER**(p, COLUMN, FIXED): col BS = 256
- **SPLITTER**(p, COLUMN, GRAIN): col BS = p
- **SPLITTER**(p, BLOCK, THREADS): s x t blocks
- **SPLITTER**(p, BLOCK, FIXED): BS = 256
- **SPLITTER**(p, BLOCK, GRAIN): BS = p
- **NOSPLIT**(): sequential execution

```
1 || PARFORBLOCK2D(iter, m, n, SPLITTER()),
2 || fgemm(..., A+iter.ibegin()*lda, lda,
3 || B+iter.jbegin(), ldb, beta,
4 || C + iter.ibegin() * ldc + iter.jbegin(), ldc);
5 ||
```
Parallelization of FFLAS-FFPACK library

Iterative cutting strategies 2D

Data parallelism: SPLITTER keyword

- SPLITTER\((p, \text{ROW, THREADS})\): \(p\) row blocks
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- NOSPLIT\(): sequential execution

\[
\begin{align*}
1 &|| \quad \text{PARFORBLOCK2D}(\text{iter}, m, n, \text{SPLITTER}()) , \\
2 &|| \quad \text{fgemm}( \ldots , A+\text{iter.ibegin()}*\text{lda} , \text{lda} , \\
3 &|| \quad B+\text{iter.jbegin()} , \text{ldb} , \text{beta} , \\
4 &|| \quad C+\text{iter.ibegin()}*\text{ldc}+\text{iter.jbegin()} , \text{ldc} ) ; \\
5 &|| \quad ) ;
\end{align*}
\]
Parallelization of FFLAS-FFPACK library

**Task parallelism**

**fork-join model:**

- LU(A11)
- ApplyP FTRSM (A12)
- ApplyP FTRSM (A21)
- ApplyP FTRSM (A13)
- ApplyP FTRSM (A31)
- FGEMM (A32)
- FGEMM (A22)
- FGEMM (A23)
- FGEMM (A33)

Time

**data-flow model:**

- LU(A11)
- ApplyP FTRSM (A12)
- ApplyP FTRSM (A13)
- ApplyP FTRSM (A31)
- FGEMM (A32)
- FGEMM (A22)
- FGEMM (A23)
- FGEMM (A33)

Time

Waiting for all tasks...
Task parallelization: fork-join and dataflow models

- **PAR_BLOCK**: opens a parallel region.
- **SYNCH_GROUP**: Group of tasks synchronized upon exit.
- **TASK**: creates a task.
  - **REFERENCE(args...)**: specify variables captured by reference. By default all variables accessed by value.
  - **READ(args...)**: set var. that are read only.
  - **WRITE(args...)**: set var. that are written only.
  - **READWRITE(args...)**: set var. that are read then written.

Example:

```c
void axpy ( const Element a , const Element b , Element &y )
{
    y += a * x ;
}
SYNCH_GROUP(
    TASK(MODE(READ(a, x) READWRITE(y)) ,
         axpy(a, x, y)) ) ;
```

Now we have a language to test our parallel exact linear algebra algorithms!
Task parallelization: fork-join and dataflow models

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**Example**:

1. ```
   void axpy(const Element a, const Element b, Element &y)
   { y += a*x; }
```

2. ```
   SYNCH_GROUP(
   TASK(MODE(READ(a, x) READWRITE(y)),
       axpy(a, x, y));
   )
   ```
PALADIn description: task parallelism

Task parallelization: fork-join and dataflow models

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Example:

```c
1  void axpy(const Element a, const Element b, Element &y){ y += a*x; }
2  SYNCH_GROUP(
3     TASK(MODE(READ(a, x) READWRITE(y))),
4       axpy(a, x, y));
5  ) ;
```

Now we have a language to test our parallel exact linear algebra algorithms!
Parallel matrix multiplication cascading

**Algorithms**
- Classical algorithms: $O(n^3)$
- Fast algorithms: $O(n^\omega)$

**Problem**
What are the best possible cascades?

**Cascading**
- Parallel classical variant switches to:
  - sequential fast
  - sequential classical
  - parallel fast
- iterative (BLOCK-THREADS)
- recursive (1D, 2D, 3D splitting)
**Parallel matrix multiplication cascading**

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- recursive (1D, 2D, 3D splitting)
- recursive (Strassen-Winograd)
Motivation

Parallel Building Blocks

Parallel matrix multiplication cascading

Algorithms

- Classical algorithms: $O(n^3)$
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Problem

What are the best possible cascades?

Cascading

- Parallel classical variant switches to:
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- Parallel fast variant switches to:
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  - parallel classical

- iterative (BLOCK-THREADS)
- recursive (1D, 2D, 3D splitting)
- recursive (Strassen-Winograd)
Performance of pfgemm

pfgemm : Parallel classical variant → Sequential fast

pfgemm on 32 cores Xeon E4620 2.2Ghz with OpenMP

**Figure**: Speed of different matrix multiplication cutting strategies using OpenMP tasks
**Performance of pfgemm**

![Graph showing performance of pfgemm on 32 cores Xeon E4620 2.2Ghz with TBB](image)

**Figure**: Speed of different matrix multiplication cutting strategies using TBB tasks
**Performance of pfgemm**

![Graph showing the performance of pfgemm on 32 cores Xeon E4620 2.2Ghz with libkomp]

**Figure**: Speed of different matrix multiplication cutting strategies using xKaapi tasks
Parallel Matrix Multiplication: State of the art

HPAC server: 32 cores Xeon E4620 2.2Ghz (4 NUMA sockets)

Comparison of our best implementations with the state of the art numerical libraries:

- MKL dgemm
- OpenBlas dgemm
- PLASMA-QUARK dgemm
- BensonBallard (Strassen)
Parallel Matrix Multiplication: State of the art

HPAC server: 32 cores Xeon E4620 2.2Ghz (4 NUMA sockets)

Effective Gfops = \( \frac{\text{# of field ops using classic matrix product}}{\text{time}} \).

Comparison of our best implementations with the state of the art numerical libraries:

- n^3 peak performance on 32 cores
- WinogradPar->classicPar<double>
- ClassicPar->WinogradSeq<double>
- MKL dgemm
- OpenBlas dgemm
- PLASMA-QUARK dgemm
- BensonBallard (Strassen)
Outline

1. Pivoting and rank profiles
2. Generic parallel Linear Algebra
3. Parallel exact Gaussian elimination
Gaussian elimination design

Reducing to MatMul: block versions

- Asymptotically faster ($O(n^\omega)$)
- Better cache efficiency

Variants of block versions

Split on one dimension:
- Row or Column slab cutting

Split on 2 dimensions:
- Tile cutting
Reducing to MatMul: block versions

→ Asymptotically faster \( (O(n^\omega)) \)
→ Better cache efficiency

Variants of block versions

Iterative:
- Static → better data mapping control
- Dataflow parallel model → less sync

Recursive:
- Adaptive
- sub-cubic complexity
- No Dataflow → more sync
Gaussian elimination design

Reducing to MatMul: block versions

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Recursive:
- Adaptive
- sub-cubic complexity
- No Dataflow → more sync
Motivation

Iterative matrix factorization

Slab iterative

Expensive costly tasks in the critical path

- Panel factorization in sequential

Rank dynamically revealed:

- Varying workload of each block op.
Tiled iterative PLUQ decomposition

→ Panel PLUQ decomposition on each slab

Slab iterative CUP to tile iterative PLUQ

- Cutting according to columns
- Creating "more parallelism": update tasks are concurrent
- Recovering rank profiles thanks to our pivoting strategies
Parallel tile recursive PLUQ algorithm

2 × 2 block splitting
Parallel tile recursive PLUQ algorithm

Recursive call
Parallel tile recursive PLUQ algorithm

\[
pTRSM : B \leftarrow BU^{-1}
\]

1  || TASK(MODE(READ(A) READWRITE(B))),
2  || pfttrsm(..., A, lda, B, ldb));
Parallel tile recursive PLUQ algorithm

\[
p_{\text{TRSM}}: B \leftarrow L^{-1}B
\]

1 \ Task \ TASK(\text{MODE}(\text{READ}(A) \ \text{READWRITE}(B)),
2 \ Task \ pftrsm(\ldots, A, \text{lda}, B, \text{ldb}));
Parallel tile recursive PLUQ algorithm

\[
p\text{fgemm} : C \leftarrow C - A \times B
\]

1 || TASK(MODE(READ(A, B) READWRITE(C)) ,
2 || p\text{fgemm}(\ldots, A, \text{lda}, B, \text{ldb}) ;

Parallel tile recursive PLUQ algorithm

\[ pf\text{gemm}: C \leftarrow C - A \times B \]

1 || TASK (MODE (READ (A, B), READWRITE (C)),
2 || pf\text{gemm} (\ldots, A, lda, B, ldb));
Parallel tile recursive PLUQ algorithm

\[ \text{pfgemm: } C \leftarrow C - A \times B \]

1 || TASK(MODE(READ(A, B) READWRITE(C)),
2 || pfgemm(\ldots, A, lda, B, ldb));
Parallel tile recursive PLUQ algorithm

2 independent recursive calls (concurrent → tasks)

1 || TASK(MODE(READWRITE(A)) ,
2 || ppluq(..., A, lda));
Parallel tile recursive PLUQ algorithm

\[ \text{pTRSM} : B \leftarrow BU^{-1} \]

1 || TASK(MODE(READ(A) READWRITE(B))) ,
2 || pftsrn(\ldots, A, lda, B, ldb));
Parallel tile recursive PLUQ algorithm

\[ \text{pTRSM} : B \leftarrow L^{-1}B \]

1 || TASK(MODE(READ(A), READWRITE(B)),
2 || pfttrs(. . . , A, lda, B, ldb));
Parallel tile recursive PLUQ algorithm

\[
pf\text{gemm} : C \leftarrow C - A \times B
\]

1 || TASK(MODE(READ(A,B) READWRITE(C))),
2 || pf\text{gemm} (... A, lda, B, ldb));
pfgemm : $C \leftarrow C - A \times B$

1 || TASK(MODE(READ(A, B) READWRITE(C)));
2 || pfgemm(\ldots, A, lda, B, ldb);
Recursive matrix factorization

Parallel tile recursive PLUQ algorithm

\[ \text{pfgemm : } C \leftarrow C - A \times B \]

1 \| TASK(MODE(READ(A,B) READWRITE(C))),
2 \| pfgemm(..., A, lda, B, ldb));
Parallel tile recursive PLUQ algorithm

Recursive call
Parallel tile recursive PLUQ algorithm

Puzzle game (block permutations)
Tile rec : better data locality and more square blocks for M.M.
State of the art: exact vs numerical linear algebra

State of the art comparison:
- Exact PLUQ using PALADIn language: best performance with xKaapi
- Numerical LU (dgetrf) of PLASMA-Quark and MKL dgetrf

Parallel dgetrf vs parallel PLUQ on full rank matrices
Performance of parallel PLUQ decomposition

Low impact of modular reductions in parallel
→ Efficient SIMD implementation

Performance of tile PLUQ recursive vs iterative on full rank matrices
### Modular reductions

<table>
<thead>
<tr>
<th>Method</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iterative Right looking</td>
<td>( \frac{1}{3} n^3 - \frac{1}{3} n )</td>
</tr>
<tr>
<td>Iterative Left Looking</td>
<td>( \frac{3}{2} n^2 - \frac{5}{2} n + 1 )</td>
</tr>
<tr>
<td>Iterative Crout</td>
<td>( \frac{3}{2} n^2 - \frac{5}{2} n + 1 )</td>
</tr>
<tr>
<td>Tile Recursive</td>
<td>( 2n^2 - n \log_2 n - 2n )</td>
</tr>
<tr>
<td>Slab Recursive</td>
<td>( (1 + \frac{1}{4} \log_2 n)n^2 - \frac{1}{2} n \log_2 n - n )</td>
</tr>
</tbody>
</table>

**Table**: Counting modular reductions in full rank LU factorization of an \( n \times n \) matrix modulo \( p \) when \( n(p - 1)^2 < 2^{\text{mantissa}} \).
Parallel Performance

Performance of task parallelism: dataflow model

Performance of tile PLUQ recursive vs iterative on full rank matrices

Possible improvement: implementation of the delegation of recursive tasks dependencies (Postpone access mode in the parallel programming environments)
Parallel Performance

**Performance of task parallelism : dataflow model**

![Graph showing performance comparison between explicit and dataflow synch PLUQ methods](image)

- **Performance of tile PLUQ recursive vs iterative on full rank matrices**
  - Explicit synch PLUQ rec<131071>
  - Dataflow synch PLUQ iter<131071>
  - Explicit synch PLUQ iter<131071>

Possible improvement: implementation of the delegation of recursive tasks dependencies (Postpone access mode in the parallel programming environments)
Performance of task parallelism: dataflow model

Possible improvement: implementation of the delegation of recursive tasks dependencies (Postpone access mode in the parallel programming environments)
## HPAC DLP challenge

- $\sim 8$ years $\rightarrow$ today feasible in $\sim 3$ months on 32 cores.

### Defended theses

- **Sub-cubic**: scale up in parallel in practice.
- **PALADIn**: parallel programming environments as a plugin
- **The rank profile matrix**: global information - efficient algorithms
  - Requires deep and precise understanding of pivoting

### Perspectives

- Study the scaling of sub-cubic exact linear algebra algorithms on distributed machines.
- PALADIn on GPUs and distributed memory machines
- Adapt Communication avoiding algorithms to compute the rank profile information